

A REMARK ON THE FAILURE OF MULTIPLICITY ONE FOR $\mathrm{GSp}(4)$

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To the memory of Ilya Piatetski-Shapiro

ABSTRACT. We revisit a classical result of Howe and Piatetski-Shapiro on the failure of strong multiplicity one for $\mathrm{GSp}(4)$.

1. INTRODUCTION

Let F be a number field. The strong multiplicity one theorem for cuspidal automorphic representations of $\mathrm{GL}(n)$ due to Piatetski-Shapiro [Pia] states that if $\pi_1 = \otimes_v \pi_{1,v}$ and $\pi_2 = \otimes_v \pi_{2,v}$ are cuspidal automorphic representations of $\mathrm{GL}(n)$ such that for all but finitely many v we have $\pi_{1,v} \cong \pi_{2,v}$, then $\pi_1 = \pi_2$. In his proof Piatetski-Shapiro used the uniqueness of the Whittaker model. The $n = 2$ case of this theorem was proved by Casselman [C].

The strong multiplicity one theorem does not hold for other classical groups, such as symplectic groups. When two representations π_1 and π_2 have the property that $\pi_{1,v} \cong \pi_{2,v}$ for all v outside of a finite set of places of F , they are said to be *nearly equivalent*. Howe and Piatetski-Shapiro [H-PS] constructed examples of nearly equivalent representations for $\mathrm{Sp}(4)$ that are not isomorphic. Cogdell and Piatetski-Shapiro showed that for any positive integer n , there exist inequivalent cuspidal automorphic representations π_1, \dots, π_n of $\mathrm{PGSp}(4)$ that are nearly equivalent [C-PS]. However, if π_1 and π_2 are nearly equivalent generic automorphic representations for $\mathrm{GSp}(4)$, then Soudry [So] showed that $\pi_1 = \pi_2$.

Let K/F be a quadratic extension, and let T be a torus in $\mathrm{GL}(2)$ F -isomorphic to K^\times . Let χ be a grossencharacter of K . The purpose of this note is to prove the following theorem:

Theorem 1. *Let F be a totally real number field, and let S be a non-empty finite set of inert primes of K/F of even cardinality. Then there are two automorphic cuspidal representations of $\mathrm{GSp}(4)$ $\pi = \otimes_v \pi_v, \pi' = \otimes_v \pi'_v$ such that*

- π is generic, but has no (T, χ) -Bessel model;
- π' is not generic, but has a (T, χ) -Bessel model;
- for all $v \notin S$, $\pi_v \simeq \pi'_v$;
- for all $v \in S$, $\pi_v \not\simeq \pi'_v$.

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The restriction on the set S is to guarantee the existence of a non-split quaternion algebra D over F containing K ramified precisely at the places in S . We fix this quaternion algebra throughout the paper.

As in the classical paper of [H-PS] the proof of this theorem uses theta correspondence from two different orthogonal groups. Here, however, Bessel coefficients, locally and globally, play a prominent role. One of the main difficulties in [H-PS] is constructing representations on a non-split orthogonal group whose theta lift to $\mathrm{GSp}(4)$ is non-zero. We use a globalization theorem of Prasad and Schulze-Pillot to construct certain cuspidal representations on the non-split orthogonal group. Using Bessel coefficients we prove that these representations have non-zero theta lift to $\mathrm{GSp}(4)$.

That the strong multiplicity one theorem for $\mathrm{GSp}(4)$ fails is of course well-known, see e.g. [H-PS], [R2]. The contribution of this modest note, if any, is to show how easily multiplicity one fails and how prevalent this phenomena is.

A bit of notation. In this paper $\mathrm{GSp}(4)$ is the group of similitude transformations of a four dimensional symplectic space. If D is a quaternion algebra, the Jacquet-Langlands transfer of a representation π of D^\times , locally and globally, is denoted by π^{JL} . If π is a representation of $\mathrm{GL}(2)$, again locally and globally, its Jacquet-Langlands transfer to D^\times is denoted by π_{JL} . If π is not square integrable, we define $\pi_{JL} = \{0\}$.

This paper is organized as follows. Section 2 contains preliminaries. The proof of the theorem is presented in Section 3.

2. BACKGROUND AND PRELIMINARIES

2.1. Orthogonal groups. Let V be the vector space M_2 , of the two by two matrices, equipped with the quadratic form \det . Let $(,)$ be the associated non-degenerate inner product, and $H = \mathrm{GO}(V, (,))$ be the group of orthogonal similitudes of V , $(,)$. The group $\mathrm{GL}(2) \times \mathrm{GL}(2)$ has a natural involution t defined by $t(g_1, g_2) = ({}^t g_2^{-1}, {}^t g_1^{-1})$, where the superscript t stands for the transposition. Let $\tilde{H} = (\mathrm{GL}(2) \times \mathrm{GL}(2)) \rtimes \langle t \rangle$ be the semi-direct product of $\mathrm{GL}(2) \times \mathrm{GL}(2)$ with the group of order two generated by t . There is a sequence

$$(1) \quad 1 \longrightarrow \mathbb{G}_m \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1,$$

where the homomorphism $\rho : \tilde{H} \rightarrow H$ is defined by $\rho(g_1, g_2)(v) = g_1 v g_2^{-1}$, and $\rho(t)v = {}^t v$, for all $g_1, g_2 \in \mathrm{GL}(2)$ and $v \in V$. Also, $\mathbb{G}_m \rightarrow \tilde{H}$ is the natural map $z \mapsto (z, z) \times 1$. It follows that the image of the subgroup $\mathrm{GL}(2) \times \mathrm{GL}(2) \subset \tilde{H}$ under ρ is the connected component of the identity of H . We usually denote H by $\mathrm{GO}(2, 2)$.

Similarly, if D is a non-split quaternion algebra we may repeat the above discussion with the reduced norm instead of \det . The corresponding general orthogonal group is denoted by $\mathrm{GO}(4)$.

Let X be four dimensional and either split or anisotropic. Then a representation π of $\mathrm{GSO}(X)$ over a local field is given by a pair of representations π_1, π_2 of either

$\mathrm{GL}(2)$ or D^\times with the same central character. If $\pi_1 \not\cong \pi_2$, the π is called *regular*; otherwise, we call it *invariant*. If π is regular, then

$$\mathrm{ind}_{\mathrm{GSO}(X)}^{\mathrm{GO}(X)} \pi$$

is irreducible and is denote by $(\pi_1, \pi_2)^+$. We also formally set $(\pi_1, \pi_2)^- = \{0\}$. In contrast, if π is invariant, this induction is reducible and has two irreducible pieces (π_1^+, π_2) , $(\pi_1, \pi_2)^-$. There is a canonical characterization of the representations $(\pi_1, \pi_2)^\pm$. Pick an anisotropic vector $y \in X$ and let Y be its orthogonal complement. The stabilizer of y in $\mathrm{O}(X)$ is $\mathrm{O}(Y)$. We say π is distinguished if

$$\mathrm{Hom}_{\mathrm{SO}(Y)}(\pi, 1) \neq 0.$$

If π is distinguished then the above Hom space is in fact one dimensional. Then

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{O}(Y)}(\pi^+, 1) + \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{O}(Y)}(\pi^-, 1) = 1.$$

By definition π^+ is the representation that affords the non-zero functional. Equivalent characterizations of π^\pm appear in [H-PS]. Then in the cases of our interest an invariant representation is distinguished.

There is an explicit transfer of automorphic representations, a Jacquet-Langlands transfer (“ JL ”) from $\mathrm{GO}(4)$ to $\mathrm{GO}(2, 2)$. We follow §7 of [H-PS] where the map is defined to be

$$(\pi_1, \pi_2)^\pm \mapsto (\pi_1^{JL}, \pi_2^{JL})^\pm,$$

locally. The global map is constructed by patching local maps together.

2.2. Theta correspondence. Our reference for theta correspondence for dual pairs $(\mathrm{GO}(X), \mathrm{GSp}(W))$ is [H-K]. For the particular case of our interest where $\dim X = \dim W = 4$ the local non-archimedean theory has been worked out in [R1]. We give a brief summary of this theory.

The main theorem of [R1] asserts:

Theorem 2. *Let σ be an irreducible admissible representation of $\mathrm{GO}(X)$ with X a four dimensional quadratic space over a non-archimedean local field. Then σ has non-trivial theta lift to $\mathrm{GSp}(4)$ if and only if σ is not of the form π^- for some distinguished irreducible admissible representation π of $\mathrm{GSO}(X)$.*

Proof. Theorem 6.8 of [R1]. □

In the global setting the theta lift of a generic representation of $\mathrm{GO}(2, 2)$ to $\mathrm{GSp}(4)$ is non-zero if there is no local obstruction. More generally we have the following theorem:

Theorem 3. *Let π_1, π_2 be a pair of non-one dimensional automorphic cuspidal representations of D^\times or $\mathrm{GL}(2)$ over a totally real field F . Let π be an automorphic cuspidal representation of $\mathrm{GO}(2, 2)$ or $\mathrm{GO}(4)$ whose restriction to the corresponding GSO contains $\pi_1 \otimes \pi_2$. Then the theta lift of π to $\mathrm{GSp}(4)$ is non-zero if there is no local obstruction.*

Proof. See Theorem 1.3. of [Tk]. □

2.3. Bessel models. Let $S \in M_2(F)$ be a symmetric matrix. We define a subgroup T of $\mathrm{GL}(2)$ by

$$T = \{g \in \mathrm{GL}(2) \mid {}^t g S g = \det g \cdot S\}.$$

Then we consider T as a subgroup of $\mathrm{GSp}(4)$ via

$$t \mapsto \begin{pmatrix} t & \\ & \det t \cdot {}^t t^{-1} \end{pmatrix},$$

$t \in T$.

We denote by U the subgroup of $\mathrm{GSp}(4)$ defined by

$$U = \{u(X) = \begin{pmatrix} I_2 & X \\ & I_2 \end{pmatrix} \mid X = {}^t X\}.$$

Finally, we define a subgroup R of $\mathrm{GSp}(4)$ by $R = TU$.

Let ψ be a non-trivial character of $F \backslash \mathbb{A}$, and define a character ψ_S on $U(\mathbb{A})$ by $\psi_S(u(X)) = \psi(\mathrm{tr}(SX))$ for $X = {}^t X \in M_2(\mathbb{A})$. Let Λ be a character of $T(F) \backslash T(\mathbb{A})$. Denote by $\Lambda \otimes \psi_S$ the character of $R(\mathbb{A})$ defined by $(\Lambda \otimes \psi)(tu) = \Lambda(t)\psi_S(u)$ for $t \in T(\mathbb{A})$ and $u \in U(\mathbb{A})$.

Let π be an automorphic cuspidal representation of $\mathrm{GSp}_4(\mathbb{A})$ and V_π its space of automorphic functions. We assume that

$$(2) \quad \Lambda|_{\mathbb{A}^\times} = \omega_\pi.$$

Then for $\varphi \in V_\pi$, we define a function B_φ on $\mathrm{GSp}_4(\mathbb{A})$ by

$$(3) \quad B_\varphi(g) = \int_{Z_{\mathbb{A}} R_F \backslash R_{\mathbb{A}}} (\Lambda \otimes \psi_S)(r)^{-1} \cdot \varphi(rh) dh.$$

We say that π has a global Bessel model of type (T, Λ) for π if for some $\varphi \in V_\pi$, the function B_φ is non-zero. In this case, the \mathbb{C} -vector space of functions on $\mathrm{GSp}_4(\mathbb{A})$ spanned by $\{B_\varphi \mid \varphi \in V_\pi\}$ is called the space of the global Bessel model of π .

Similarly, one can consider local Bessel models. Fix a local field F . Define the algebraic groups T_S , U , and R as above. Also, consider the characters Λ , ψ , ψ_S , and $\Lambda \otimes \psi_S$ of the corresponding local groups. Let (π, V_π) be an irreducible admissible representation of the group $\mathrm{GSp}(4)$ over F . Then we say that the representation π has a local Bessel model of type (T, Λ) if there is a functional $\lambda_B \in (V_\pi^\infty)'$, a linear functional on V_π^∞ satisfying

$$\lambda_B(\pi(r)v) = (\Lambda \otimes \psi_S)(r)\lambda_B(v),$$

for all $r \in R(F)$, $v \in V_\pi$.

The fundamental properties of Bessel models in the local setting are established in [P-TB].

2.4. Tunnell dichotomy theorem. The following theorem is fundamental in this work:

Theorem 4 (Tunnell Dichotomy Theorem). *Let F be local field. Let T be a torus in $\mathrm{GL}(2)$, and λ a quasi-character of $T(F)$. For every irreducible admissible representation Π of $\mathrm{GL}_2(F)$, we have*

$$(4) \quad \dim_{\mathbb{C}} \mathrm{Hom}_{T(F)}(\Pi, \lambda) + \dim_{\mathbb{C}} \mathrm{Hom}_{T(F)}(\Pi_{JL}, \lambda) = 1.$$

Here Π_{JL} is the Jacquet-Langlands lift of Π to the unique quaternion algebra D over F . Hence, $\dim_{\mathbb{C}} \mathrm{Hom}_{T(F)}(\Pi^{JL}, \lambda) = 0$ if Π is not discrete series, or if $T(F)$ is split.

Proof. See [S] and [T]. \square

If for some representation Π , the space

$$\mathrm{Hom}_{T(F)}(\Pi, \lambda) \neq 0$$

we say that Π has a (T, χ) -Waldspurger model. There is a similar notion in the global setting.

2.5. A globalization theorem. We also recall the following globalization theorem of Prasad and Schulze-Pillot [P-S]:

Theorem 5 (Globalization Theorem). *Let H be a closed algebraic subgroup of a reductive group G both defined over a number field F . Let Z be the identity component of the center of G . Assume $Z \leq H$ and that H/Z has no F -rational characters. Let $\chi = \otimes'_v \chi_v$ be a one dimensional automorphic representation of $H(\mathbb{A})$. Suppose S is a finite set of non-archimedean places of F and for each $v \in S$ we are given an irreducible supercuspidal representation π_v such that*

$$\mathrm{Hom}_{H(F_v)}(\pi_v, \chi_v) \neq 0.$$

Let S' be a finite set of places containing S and all the infinite places, such that G is quasi-split at places outside S' , and χ_v is unramified outside S' . Then there exists an automorphic cuspidal representation $\Pi = \otimes'_v \Pi_v$ of $G(\mathbb{A})$ such that

- $\Pi_v = \pi_v$ for $v \in S$;
- Π_v is unramified at all finite places of F outside S' ; and
- there is an $f \in \Pi$ such that

$$\int_{H(F)Z(\mathbb{A}) \backslash H(\mathbb{A})} f(h) \chi(h)^{-1} dh \neq 0.$$

Proof. See Theorem 4.1 of [P-S]. \square

We will apply this theorem in the following manner. Let T be a torus embedded in the multiplicative group of a non-split quaternion algebra D defined over a number field F . Let $\chi = \otimes'_v \chi_v$ be an one dimensional automorphic representation of T . Since T is anisotropic, T/Z does not have any F -rational characters. Let v be a place where T is split, and let π_v be a supercuspidal representation of $D^\times(F_v) \simeq \mathrm{GL}_2(F_v)$. Then it follows from Theorem 4 that

$$\mathrm{Hom}_{T(F_v)}(\pi_v, \chi_v) \neq 0.$$

Let S' be the finite set of places consisting of places in S , archimedean places, places where χ ramifies, and v . Then the theorem asserts that there is an automorphic cuspidal representation $\Pi = \otimes'_v \Pi_v$ of $D^\times(\mathbb{A})$ unramified outside S' such that $\Pi_v \simeq \pi_v$ and Π has a global (T, χ) -Waldspurger model.

3. PROOF OF THE THEOREM

As in [H-PS] we will use the following diagram

$$\begin{array}{ccc}
 & \text{GO}(2,2) & \\
 & \uparrow \scriptstyle JL & \searrow \scriptstyle \theta \\
 & \vdots & \text{GSp}(4) \\
 & \downarrow & \nearrow \scriptstyle \theta \\
 & \text{GO}(4) &
 \end{array}$$

Here $\text{GO}(4)$ is constructed using a non-split quaternion algebra D , ramified precisely at the places in S , that contains the torus T . Such a quaternion algebra exists if K/F is an extension of fields. One can then consider in the local situation the theta lift of the representations $(\pi_1, \pi_2)^+$ and $(\pi_1^{JL}, \pi_2^{JL})^+$ to $\text{GSp}(4)$; we denote these representations by $\theta(\pi_1, \pi_2)$ and $\theta(\pi_1^{JL}, \pi_2^{JL})$, respectively.

Lemma 1. *In the local situation, the representations $\theta(\pi_1, \pi_2)$ and $\theta(\pi_1^{JL}, \pi_2^{JL})$ are non-zero and are inequivalent.*

Proof. In the archimedean situation this lemma is well-known. The non-vanishing of the local theta lifts follows from Remark 6.8 of [P-TB] as follows. We will show that $\theta(\pi_1, \pi_2)$ and $\theta(\pi_1^{JL}, \pi_2^{JL})$ have some non-zero Bessel model. To see this it suffices to show that π_1, π_2 , and π_1^{JL}, π_2^{JL} respectively, have a common Waldspurger model. For $\text{GL}(2)$ representations this is obvious, as by Theorem 4 every representation has almost all Waldspurger models for a given torus. For D^\times , if $\pi_1 \simeq \pi_2$, this is obvious; if $\pi_1 \not\simeq \pi_2$, we need to show the existence of a torus whose trivial characters occurs in $\pi_1 \otimes \tilde{\pi}_2$. For this see [P]. The non-vanishing of the theta lift also follows from Theorem 2.

In order to prove that the two representations are inequivalent we will use Bessel models. Namely we will show the existence of a Bessel model for the representation $\theta(\pi_1^{JL}, \pi_2^{JL})$ that other representations cannot have. By Corollary 7.1 of [P-TB] the representation $\theta(\pi_1^{JL}, \pi_2^{JL})$ will have a (T, χ) Bessel model if and only if the two representations π_1^{JL}, π_2^{JL} have (T, χ) -Waldspurger models. It follows from Theorem 4 that if T is split, π_1^{JL} and π_2^{JL} will have (T, χ) -Waldspurger models. On the other hand, by the same corollary since π_1, π_2 do not have (T, χ) -Waldspurger models, $\theta(\pi_1, \pi_2)$ cannot have (T, χ) -Bessel models. \square

Lemma 2. *Let D be a non-split quaternion algebra over a totally real number field F which is ramified at places in a set S . Let (π_1, π_2) be a pair of non-isomorphic automorphic representations of $D^\times(\mathbb{A}_F)$, and suppose that $\theta(\pi_1, \pi_2)$ and $\theta(\pi_1^{JL}, \pi_2^{JL})$ are non-zero. Then*

- for $v \in S$, $\theta(\pi_1, \pi_2)_v \not\simeq \theta(\pi_1^{JL}, \pi_2^{JL})_v$;
- for $v \notin S$, $\theta(\pi_1, \pi_2)_v \simeq \theta(\pi_1^{JL}, \pi_2^{JL})_v$.

Proof. This is a direct consequence of the previous lemma. \square

We can now present the proof of the main theorem.

Proof of Theorem 1. By Lemma 2 it suffices to find a pair of non-equivalent automorphic representations (π_1, π_2) of $D^\times(\mathbb{A}_F)$ such that $\theta(\pi_1, \pi_2)$ and $\theta(\pi_1^{JL}, \pi_2^{JL})$

are non-zero. In order to do this we will use the discussion following Theorem 5 to find a pair (π_1, π_2) of non-equivalent representations such that both representations have a global (T, χ) -Waldspurger model. Let $v \notin S$ be a place where T_v is split. Let $\pi_{1,v}, \pi_{2,v}$ be two non-equivalent supercuspidal representations of $D_v^\times \simeq \mathrm{GL}_2(F_v)$ with the same central character as $\chi_v|_{Z_v}$. Then we know that there are automorphic representations π_1, π_2 with (T, χ) -Waldspurger models and such that their respective local components at v are $\pi_{1,v}, \pi_{2,v}$. Since $\pi_{1,v}, \pi_{2,v}$ are non-equivalent, we have $\pi_1 \not\simeq \pi_2$, and we are done. By §13 of [P-TB] the representation $\theta(\pi_1, \pi_2)$ will have a (T, χ) -Bessel model and hence cannot be zero. Note that on the other hand since $\theta(\pi_1, \pi_2)$ is non-zero, it is locally non-zero. The local component of $\theta(\pi_1, \pi_2)$ is $\theta(\pi_{1,v}, \pi_{2,v})$. This then means that $\theta(\pi_{1,v}^{JL}, \pi_{2,v}^{JL}) \neq 0$. Now since π_1^{JL}, π_2^{JL} are generic, and there is no local obstruction, the non-vanishing of global theta lift follows from Theorem 3. \square

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